Review of Energy Concepts from Strength of Materials

or some “good-to-know” background usually assumed by most vibration textbooks

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1. Try to explain why kinetic energy is $\frac{1}{2} \text{mv}^2$ ...

For a small *displacement* $\text{ds}$, the *force* on a moving particle

$= \text{m} \ \frac{dv}{dt}$

Total work done $= \int \text{m} \ \frac{dv}{dt} \ \text{ds}$

$= \int \text{m} \ \frac{ds}{dt} \ \text{dv}$

$= \text{m} \ \int \text{v} \ \text{dv}$

$= \frac{1}{2} \text{m} \ \text{v}^2$
2. Why is the elastic energy of a spring $\frac{1}{2} kx^2$?

Answer: Because...

a) Work is done by the internal force

b) The internal force is proportional to the deformation (Hooke’s Law)

$$\text{elastic energy} = \int_{0}^{x} F(s) \, ds$$

1. Deform a small amount

2. The internal force changes with deformation

3. Sum up for total deformation
For a linear spring \( F(s) = ks \)

Hence elastic energy

\[
\begin{align*}
\int_{0}^{x} k s \, ds &= \frac{1}{2} k x^2
\end{align*}
\]

Question: What happens if the spring is non-linear eg \( F(s) = ks^3 \)?
Insight

Suppose the spring has a natural length $l_0$ and a “cross-sectional area” $A$

$$U = \frac{1}{2} k x^2$$

$$= \frac{1}{2} \left( \frac{kx}{A} \right) \left( \frac{x}{l_0} \right) (A l_0)$$

stress strain volume

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3. So what is the strain energy of a deformable body?

Recall that a deformable or elastic body has

6 stress components $\sigma^T = \{ \sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx} \}$

6 strain components $\varepsilon^T = \{ \varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx} \}$

The strain energy of an elastic body is given by:

$$U = \frac{1}{2} \iiint \varepsilon^T \sigma \, dV$$

units? $V: \text{volume of body}$
Note that for a linear stress-strain law

\[ \sigma = [ D ] \varepsilon \]

Matrix of material constants

Hence

\[ U = \frac{1}{2} \iiint \varepsilon^{T} \sigma \, dV \]

\[ = \frac{1}{2} \iiint \varepsilon^{T} [ D ] \varepsilon \, dV \]

Which is the standard quadratic form!
Example: Strain energy of a bar

Displacement field is \{ u(x,t), 0, 0 \}

Only non-zero strain component \( \varepsilon_{xx} = \frac{\partial u}{\partial x} \)

Hence the only non-zero stress component is \( \sigma_{xx} = E \varepsilon_{xx} \)

Strain energy \( = \frac{1}{2} \int \int \int \varepsilon^T \sigma \, dV \)

\( = \frac{1}{2} \int \int \int \frac{\partial u}{\partial x} (E \frac{\partial u}{\partial x}) \, dV \)

\( = \frac{1}{2} \int EA (\frac{\partial u}{\partial x})^2 \, dx \)

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Example: Strain energy of a beam

Displacement field \( \{ -z \, \partial w / \partial x, \ 0, \ w(x,t) \} \)

Non zero strain \( \varepsilon_{xx} = \partial u / \partial x = -z \, \partial^2 w / \partial x^2 \)

Non zero stress \( \sigma_{xx} = -E \, z \, \partial^2 w / \partial x^2 \)

Strain energy \( = \frac{1}{2} \int \int \int \varepsilon^T \sigma \, dV \)

\[ = \frac{1}{2} \int \{ E \left( \int \int z^2 \, dz \, dy \right) \left( \partial^2 w / \partial x^2 \right)^2 \} \, dx \]

\[ = \frac{1}{2} \int EI_z \left( \partial^2 w / \partial x^2 \right)^2 \, dx \]