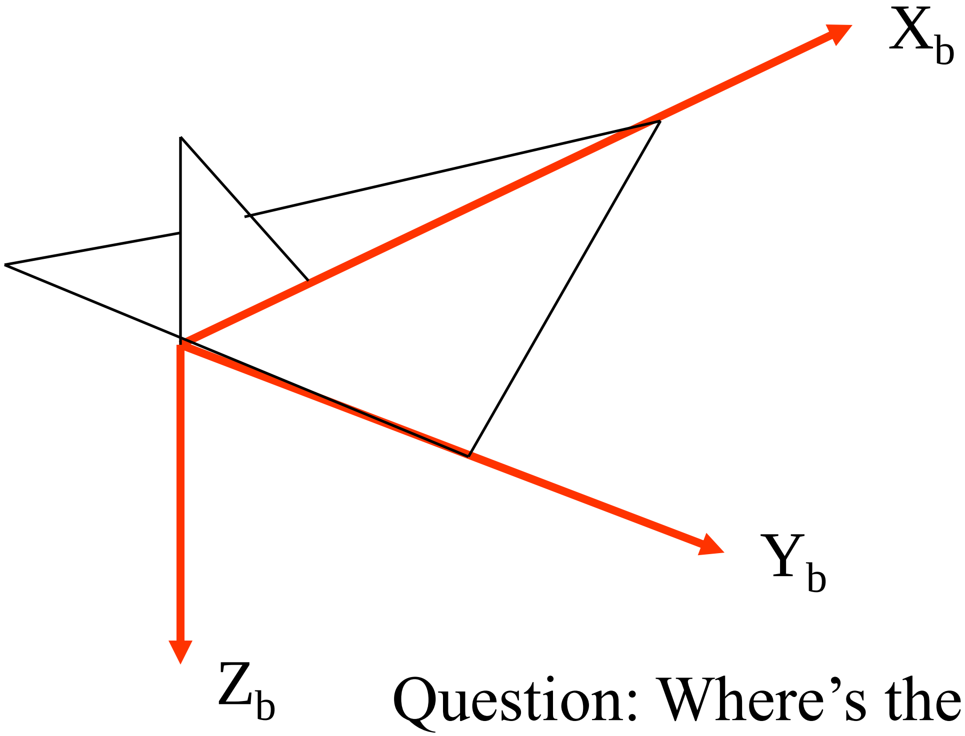
Lecture 6 : Aircraft orientation in 3 dimensions

Or describing simultaneous roll, pitch and yaw

1.0 Flight Dynamics Model

- For flight dynamics & control, the reference frame is aligned with the aircraft and moves with it. (Why?)



Question: Where's the origin located for this body axes ?

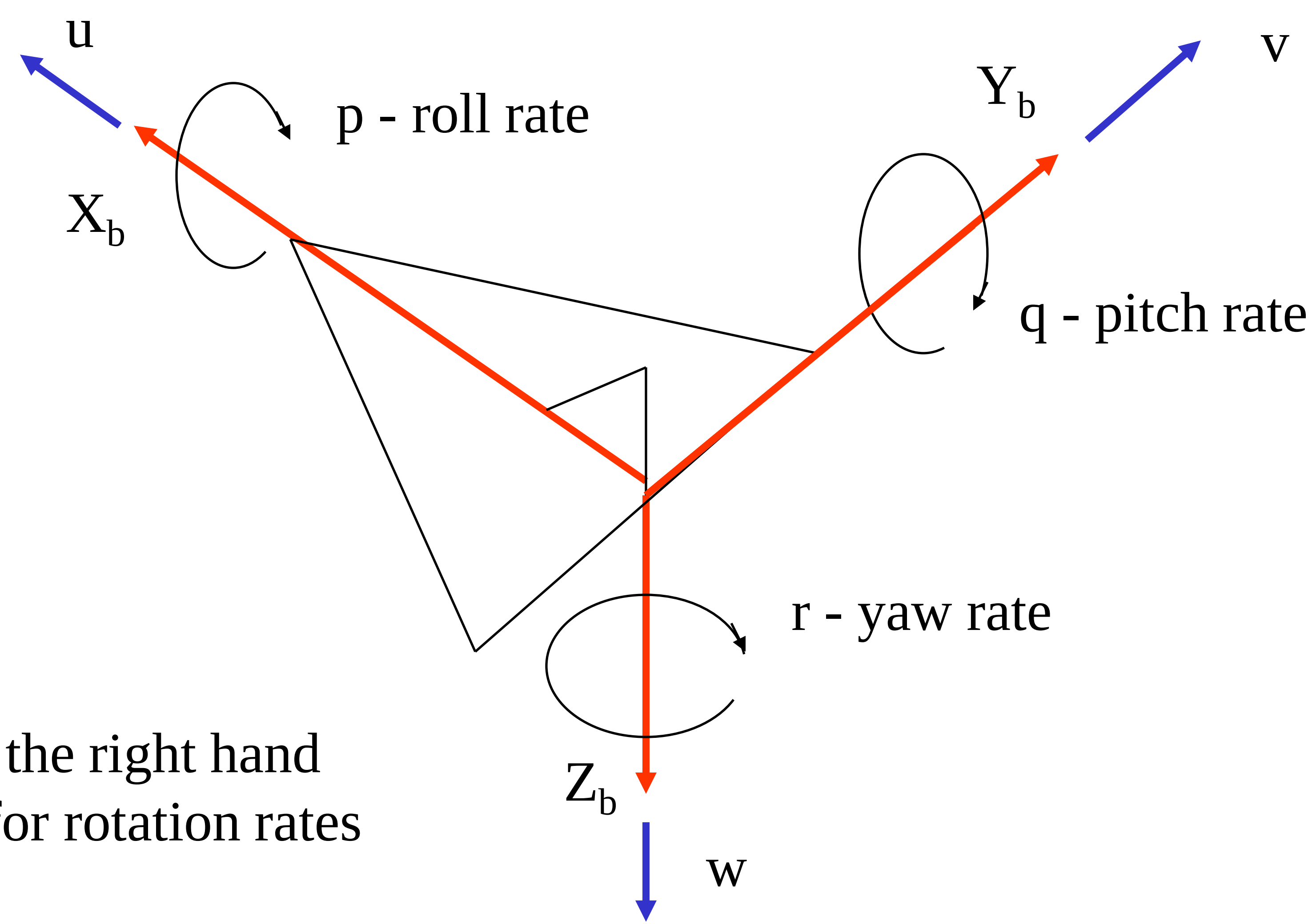
- The aircraft is modelled as a rigid body with __ degrees of freedom

- The ___ DOFs correspond to
  -
  -

- Denote the translational velocity of the aircraft by **V** = {u,v,w}

- Denote the angular velocity of the aircraft by **ω** = {p,q,r}

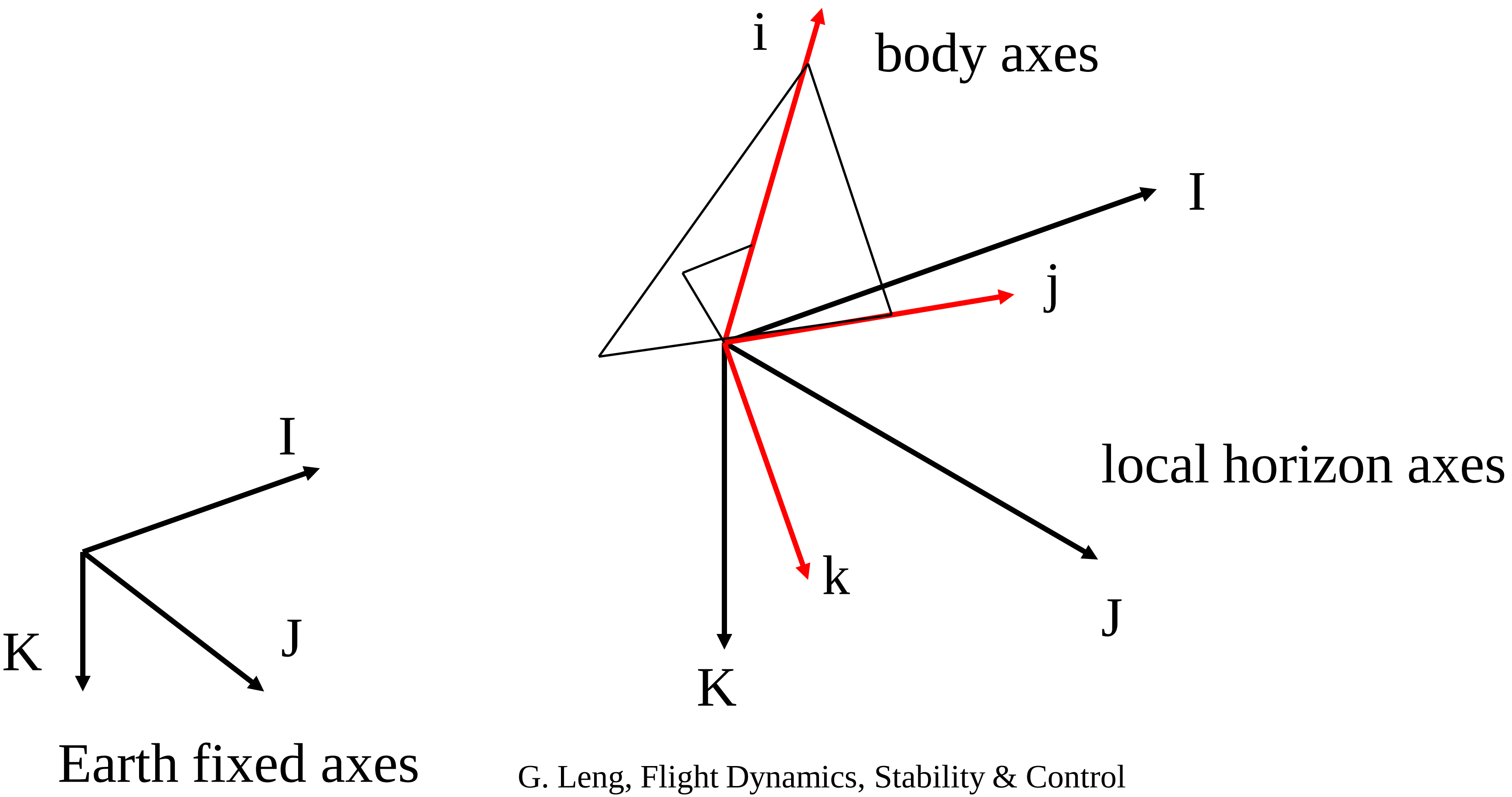
Figure 1.1 : Six Degrees of Freedom



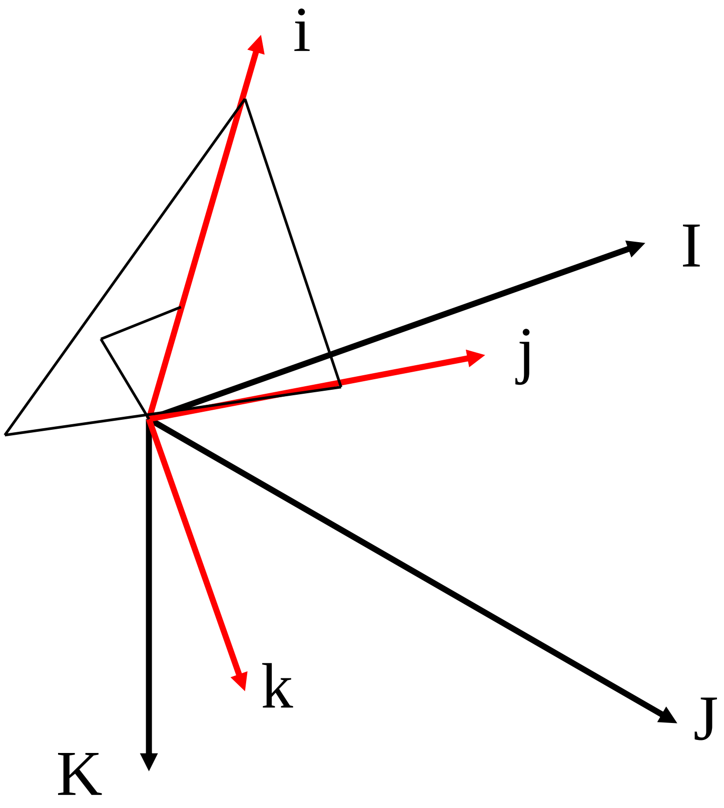
Note the right hand
rule for rotation rates

1.1 Defining the aircraft orientation - Euler angles

The local horizon axes is aligned with the Earth fixed axes but translated to the aircraft's cg.



Question : How do we express the basis vectors **IJK** of the local horizon system in terms of the basis vectors **ijk** of the body axes system or vice versa ?

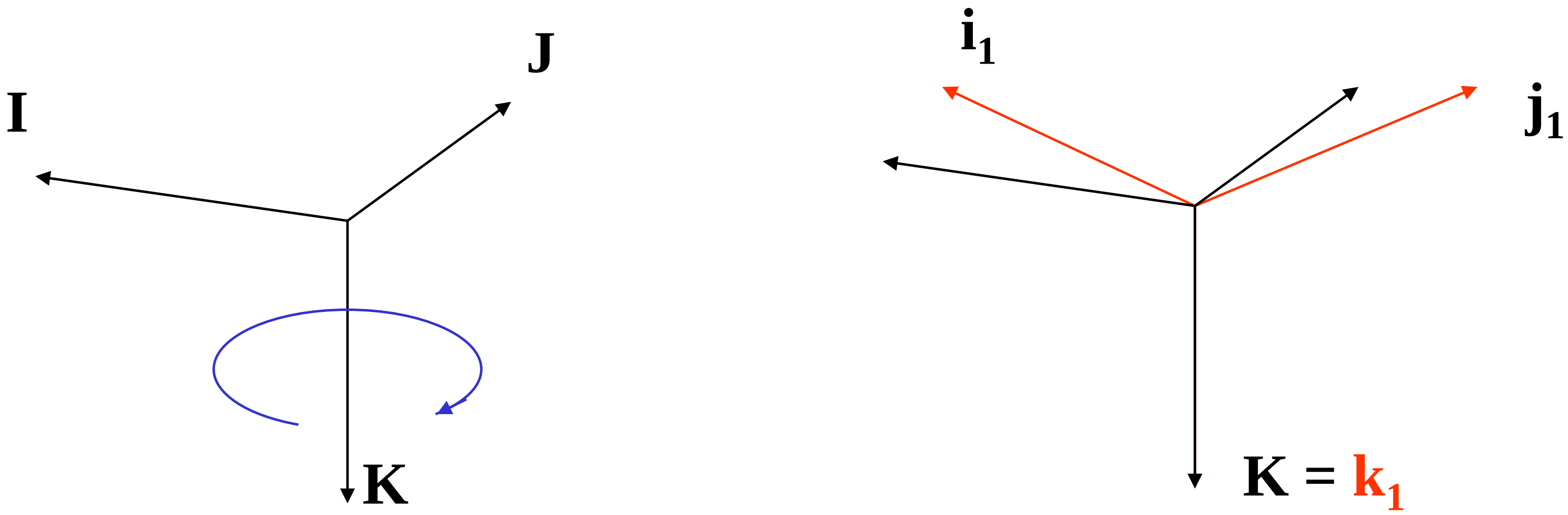


The local horizon axes system **IJK** can be rotated to coincide with the body axes **i j k** by using three rotation angles or Euler angles

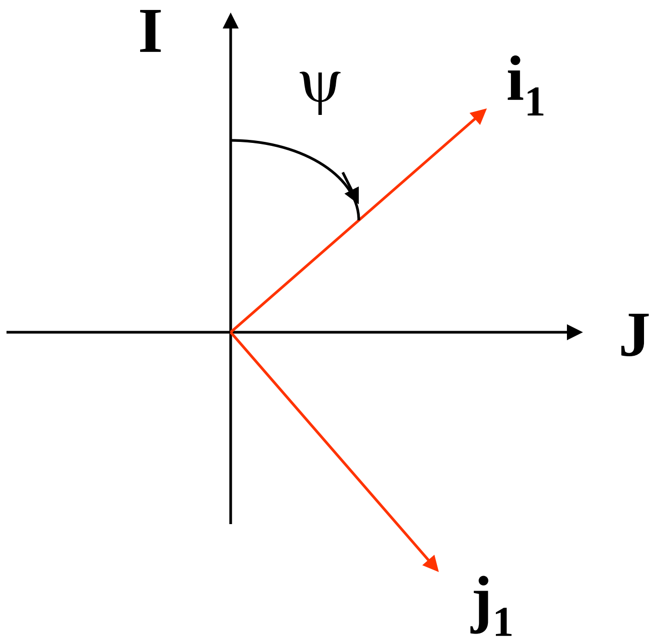
Yaw Euler angle	ψ	(Greek : psi)
Pitch Euler angle	θ	(theta)
Roll Euler angle	ϕ	(phi)

NB : The order of the rotations is important !

Step 1) Rotate **IJK** by an angle ψ about the **K** axis (yaw)
This yields the intermediate axes $\mathbf{i_1}\ \mathbf{j_1}\ \mathbf{k_1}$



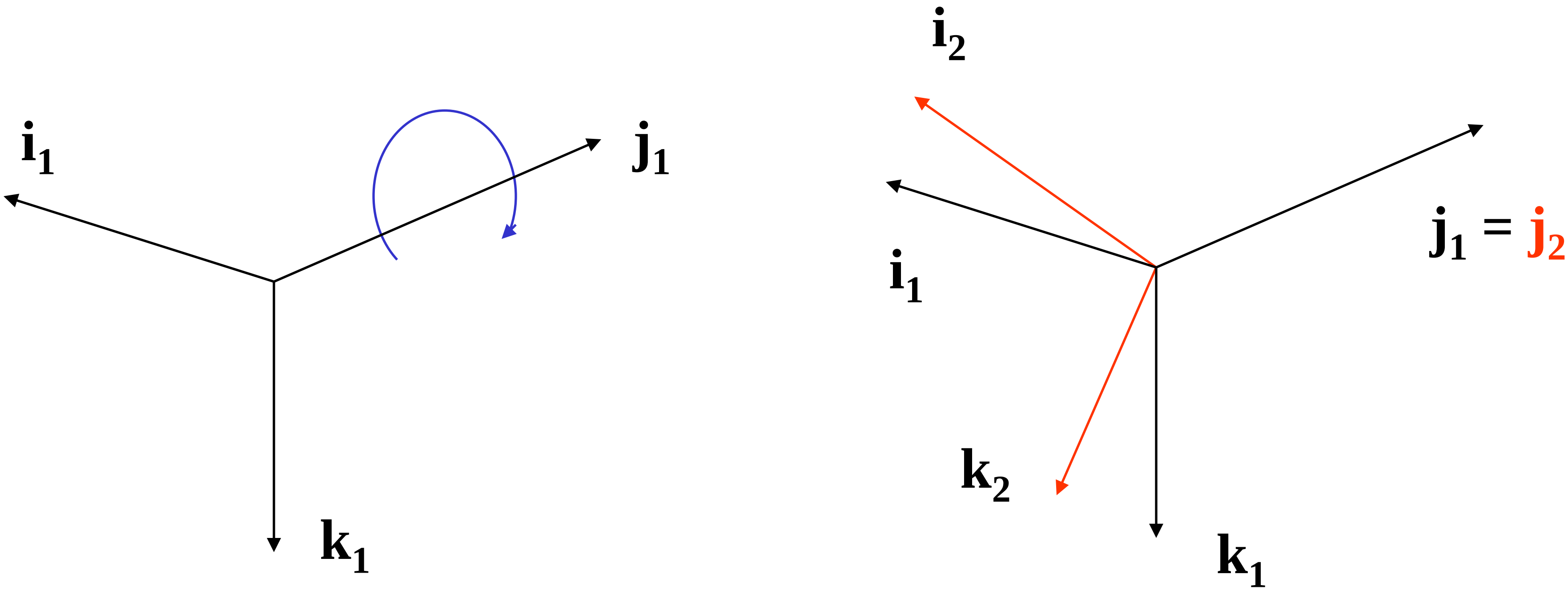
Step 1) Yaw about the K axis



$$\begin{Bmatrix} \mathbf{i_1} \\ \mathbf{j_1} \\ \mathbf{k_1} \end{Bmatrix} = [\,R_\psi\,] \begin{Bmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{Bmatrix}$$

$$= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{Bmatrix}$$

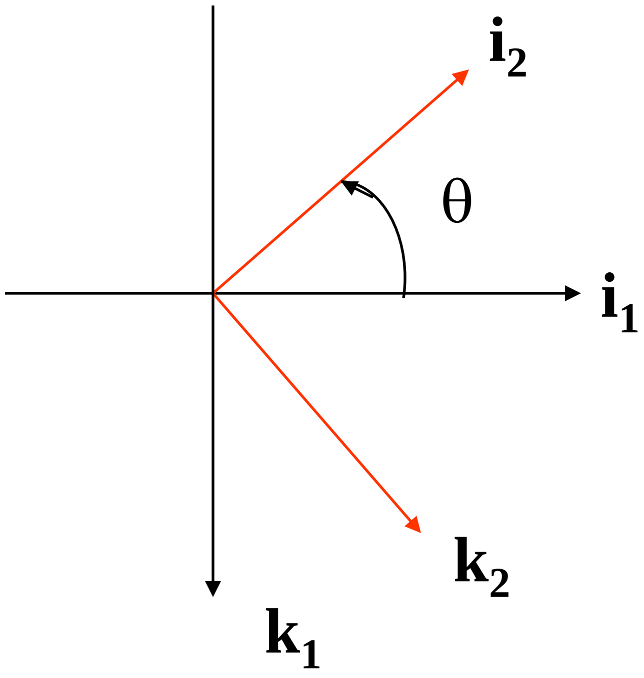
Step 2) Rotate $\mathbf{i_1\ j_1\ k_1}$ by an angle θ about the $\mathbf{j_1}$ axis (pitch)
This yields the intermediate axes $\mathbf{i_2\ j_2\ k_2}$



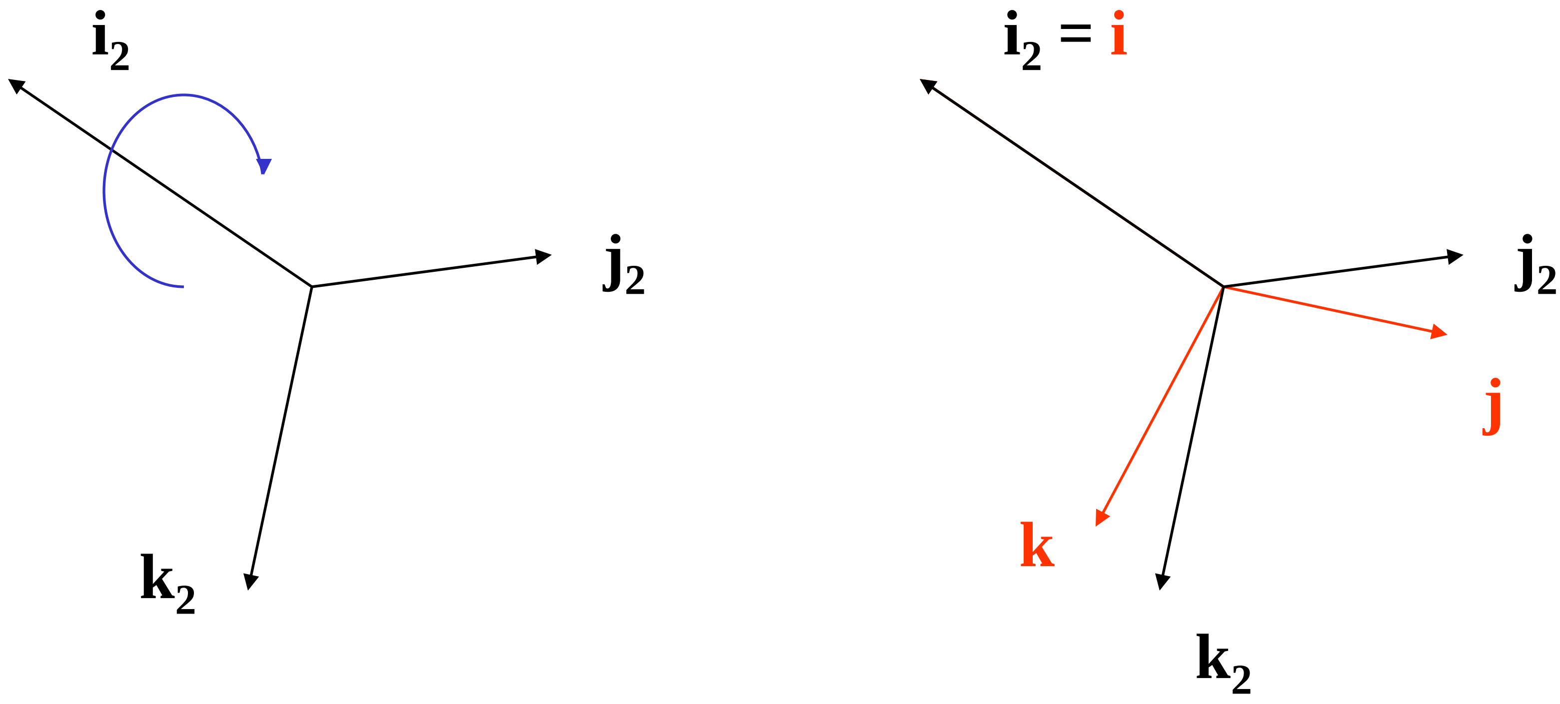
Step 2) Pitch about the j_1 axis

$$\begin{Bmatrix} \mathbf{i}_2 \\ \mathbf{j}_2 \\ \mathbf{k}_2 \end{Bmatrix} = [\,R_\theta\,] \begin{Bmatrix} \mathbf{i}_1 \\ \mathbf{j}_1 \\ \mathbf{k}_1 \end{Bmatrix}$$

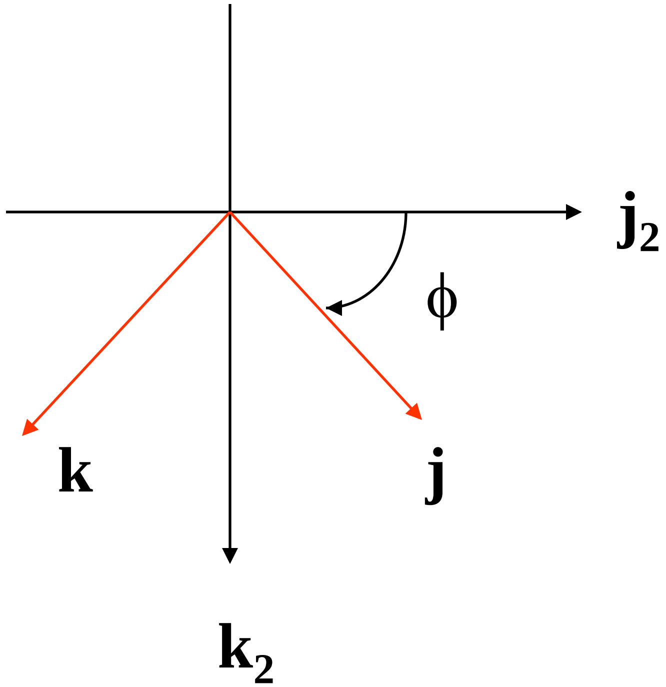
$$= \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{Bmatrix} \mathbf{i}_1 \\ \mathbf{j}_1 \\ \mathbf{k}_1 \end{Bmatrix}$$



Step 3) Rotate $\mathbf{i}_2\ \mathbf{j}_2\ \mathbf{k}_2$ by an angle ϕ about the new $\mathbf{i}_2$ axis (roll)
This yields the body axes **i j k**



Step 3) Roll about the i_2 axis



$$\begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix} = [\, R_\phi \,] \begin{Bmatrix} \mathbf{i_2} \\ \mathbf{j_2} \\ \mathbf{k_2} \end{Bmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{Bmatrix} \mathbf{i_2} \\ \mathbf{j_2} \\ \mathbf{k_2} \end{Bmatrix}$$

So finally, the basis vectors of the local horizon **IJK** and the body axes **ijk** are related as follows :

$$\begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix} = [\, R_{\phi} \,]\,[\, R_{\theta} \,]\,[\, R_{\psi} \,] \begin{Bmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{Bmatrix}$$

Question : What happens if we want **IJK** in terms of **ijk** ?

Hint : The rotation matrices have a special property

$$[\, R \,]^{-1} = [\, R \,]^{T}$$

Inverting the matrix product to get **IJK** in terms of **ijk** :

$$\begin{Bmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{Bmatrix} = \left([\,R_{\phi}\,]\,[\,R_{\theta}\,]\,[\,R_{\psi}\,] \right)^{-1} \begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix}$$

$$= [\,R_{\psi}\,]^{T}\,[\,R_{\theta}\,]^{T}\,[\,R_{\phi}\,]^{T} \begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix}$$

Exercise : Express the weight force component in terms of the body axes basis vectors **i j k**

Hint : The weight component points downwards ie W **K**

hence we need only express **K** in terms of **i j k**

Using the relationship derived :

$$\begin{Bmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{Bmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} \begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix}$$

Premultiply both sides by {0 0 1}

$$\mathbf{K} = \{-\sin\theta \;\; 0 \;\; \cos\theta\} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} \begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix}$$

$$= \{-\sin\theta \quad \cos\theta\sin\phi \quad \cos\theta\cos\phi\} \begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix}$$

What happens if the Euler angles are small ?

2.0 Rotation rates and change in aircraft orientation

How are the body rotational rates p,q,r related to the rate of change of the Euler angles ϕ θ ψ ?

How about

$$p = d\phi / dt$$

$$q = d\theta / dt$$

$$r = d\psi / dt$$

The angular velocity vector is $\boldsymbol{\omega} = p\,\mathbf{i} + q\,\mathbf{j} + r\,\mathbf{k}$ written using the body axes basis vectors

$\boldsymbol{\omega}$ describes the rate of change in orientation which can also be written as :

$$\boldsymbol{\omega} = \psi'\,\mathbf{K} + \theta'\,\mathbf{j_1} + \phi'\,\mathbf{i_2}$$

Noting that

$$\mathbf{K} = -\sin\theta\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} + \cos\theta\cos\phi\,\mathbf{k}$$

$$\mathbf{j_1} = \mathbf{j_2} = \cos\phi\,\mathbf{j} - \sin\phi\,\mathbf{k}$$

$$\mathbf{i_2} = \mathbf{i}$$

$$\begin{Bmatrix} p \\ q \\ r \end{Bmatrix} = \begin{pmatrix} -\sin\theta & 0 & 1 \\ \cos\theta \sin\phi & \cos\phi & 0 \\ \cos\theta \cos\phi & -\sin\phi & 0 \end{pmatrix} \begin{Bmatrix} \psi' \\ \theta' \\ \phi' \end{Bmatrix}$$

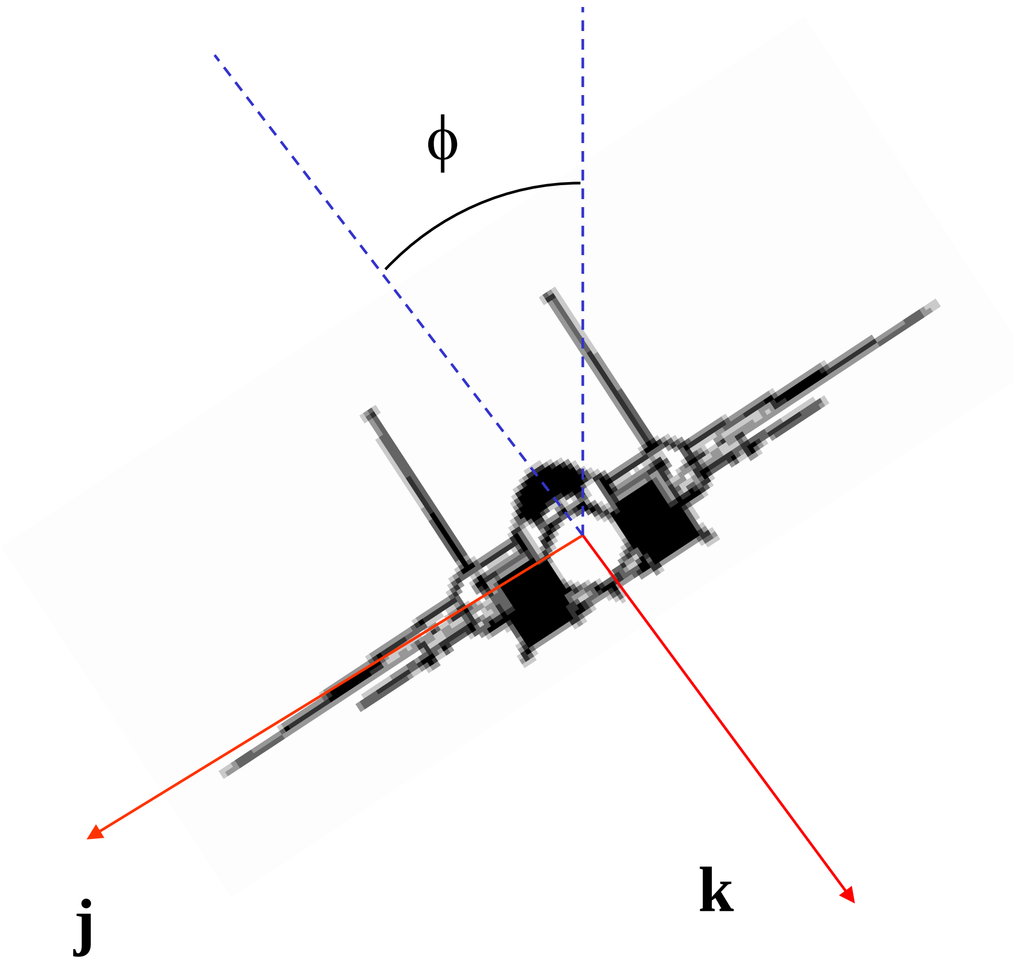
Inverting yields :

$$\phi' = p + (q \sin\phi + r \cos\phi) \tan\theta$$
$$\theta' = q \cos\phi - r \sin\phi$$
$$\psi' = (q \sin\phi + r \cos\phi) \sec\theta$$

What happens if θ is 90^{o} ?

NB : These are the EOMs relating the rate of change of aircraft orientation to body rotational rates

Ex : Rotation rates during a sustained level turn



1. What is the angular velocity vector for an aircraft executing a level sustained turn of radius R at speed V ? The bank angle is ϕ.

2. Hence state the body axes rotation rates.

Are they all zero ?

1. The turn rate = V/R

2. Level turn $\Rightarrow$ the rotation axis is vertical

3. Hence the angular velocity vector is

$$\boldsymbol{\omega} = V/R \{0, 0, 1\}$$ What is ambiguous ?

$$= V/R \, \mathbf{K}$$ not the body axes **k** !

So how do we write this in body axes ?

4. Recall the relationship derived via Euler angles

$$\mathbf{K} \quad = \quad -\sin\theta\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} + \cos\theta\cos\phi\,\mathbf{k}$$

Hence in body axes

$$\boldsymbol{\omega} \quad = \quad V/R(-\sin\theta\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} + \cos\theta\cos\phi\,\mathbf{k})$$

5. Recall $\boldsymbol{\omega}$ may be written in rotation rates as :

$$\boldsymbol{\omega} = p\,\mathbf{i} + q\,\mathbf{j} + r\,\mathbf{k}$$

Hence the body axes rotational rates for a sustained level turn are :

$$\left.\begin{aligned} p &= -V/R \sin\theta \\ q &= V/R\cos\theta \sin\phi \\ r &= V/R\cos\theta \cos\phi \end{aligned}\right\} \quad \text{Interpret this}$$

What is the implication ?

6. What happens if we substitute these sustained, level turn rotation rates

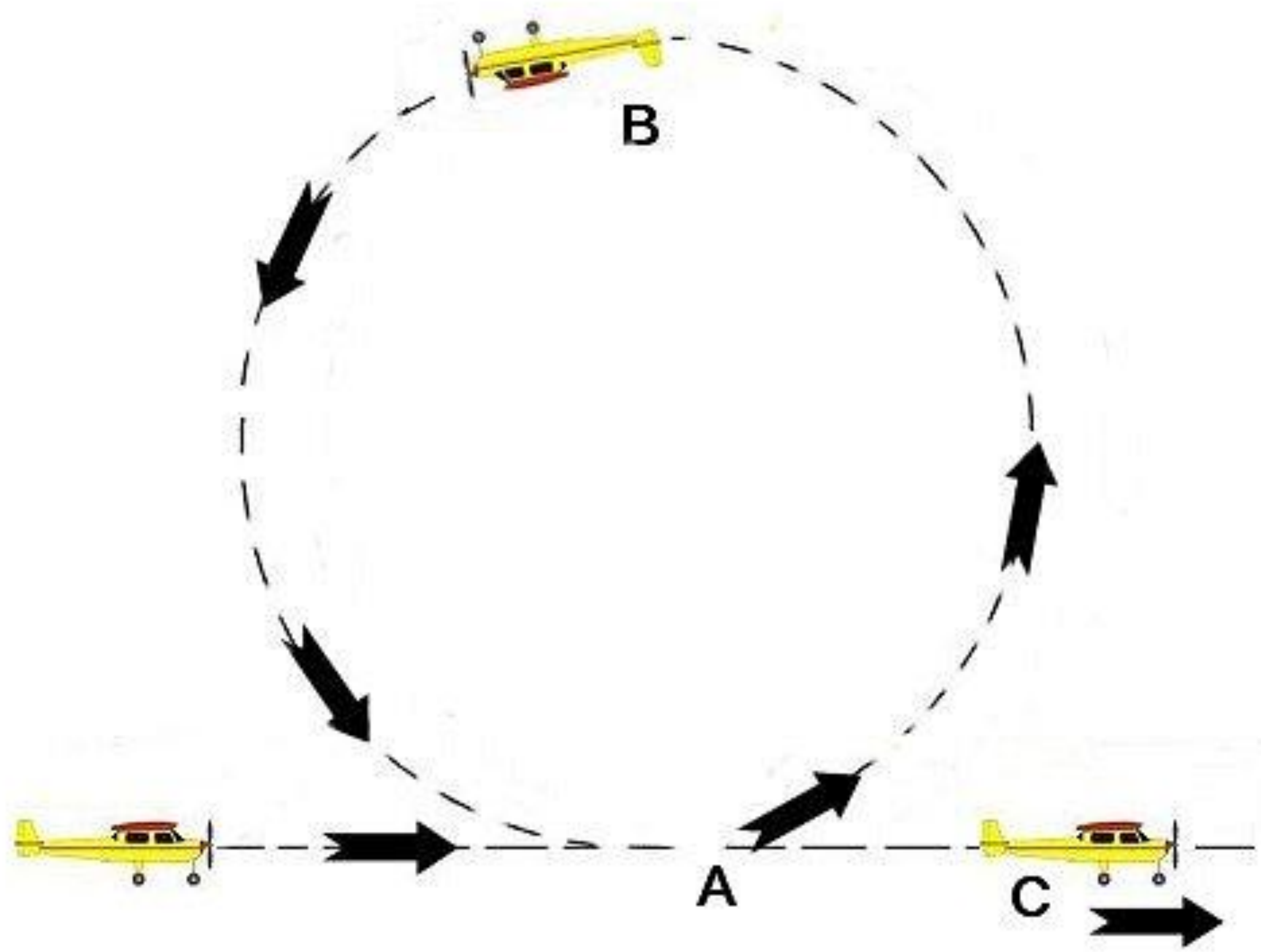
$$p = -V/R \sin\theta$$
$$q = V/R\cos\theta \sin\phi$$
$$r = V/R\cos\theta \cos\phi$$

in the equations for change in orientation ?

$$\phi' = p + (q \sin\phi + r \cos\phi)\tan\theta$$
$$\theta' = q \cos\phi - r \sin\phi$$
$$\psi' = (q \sin\phi + r \cos\phi)\sec\theta$$

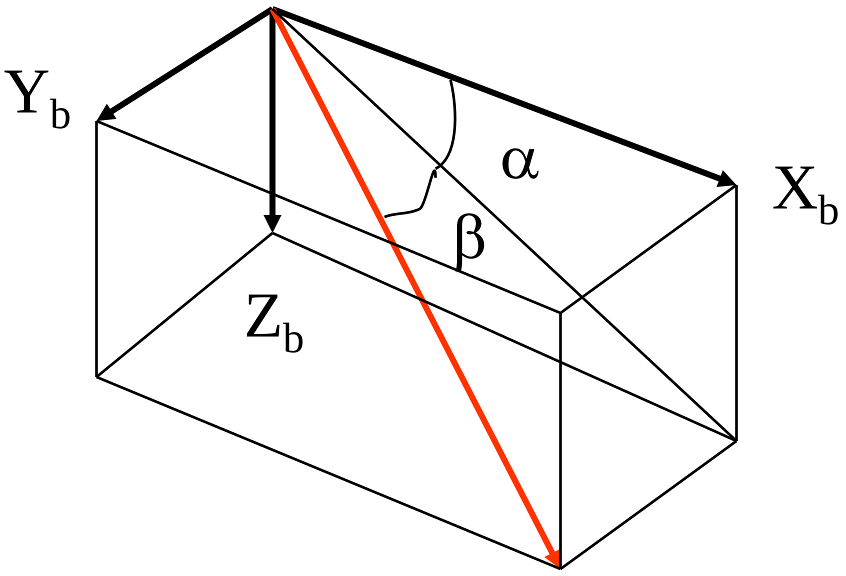
Physically what do you expect?

Follow-up exercise : Determine the rotation rates for a loop



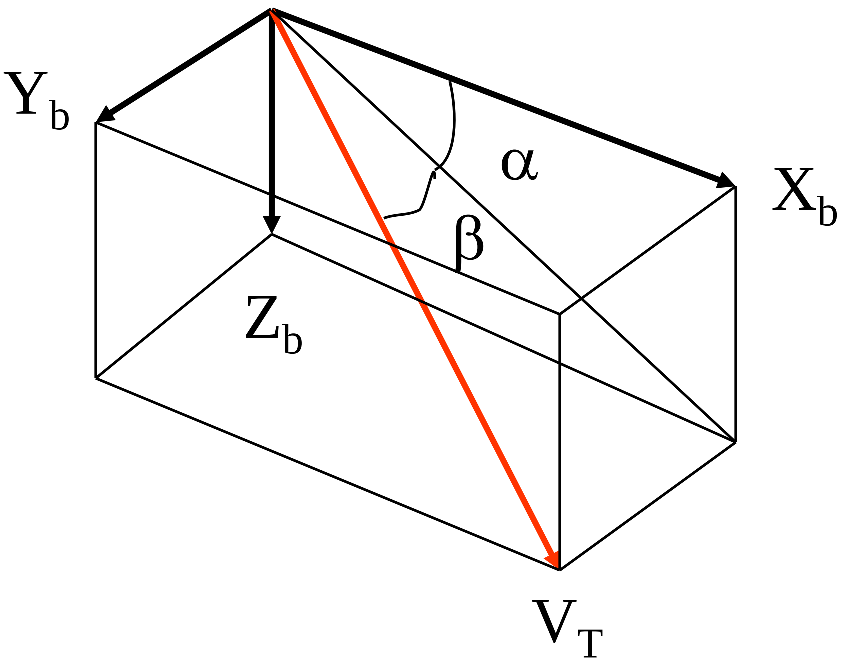
3.0 Aerodynamic forces in the body axes

For a general aircraft orientation, the angle of attack α and sideslip β are defined as follows:



V direction of flight

3.1 : Writing body velocity components with aerodynamic angles



Resolving the velocity vector

$$u = V_T \cos\beta \cos\alpha$$

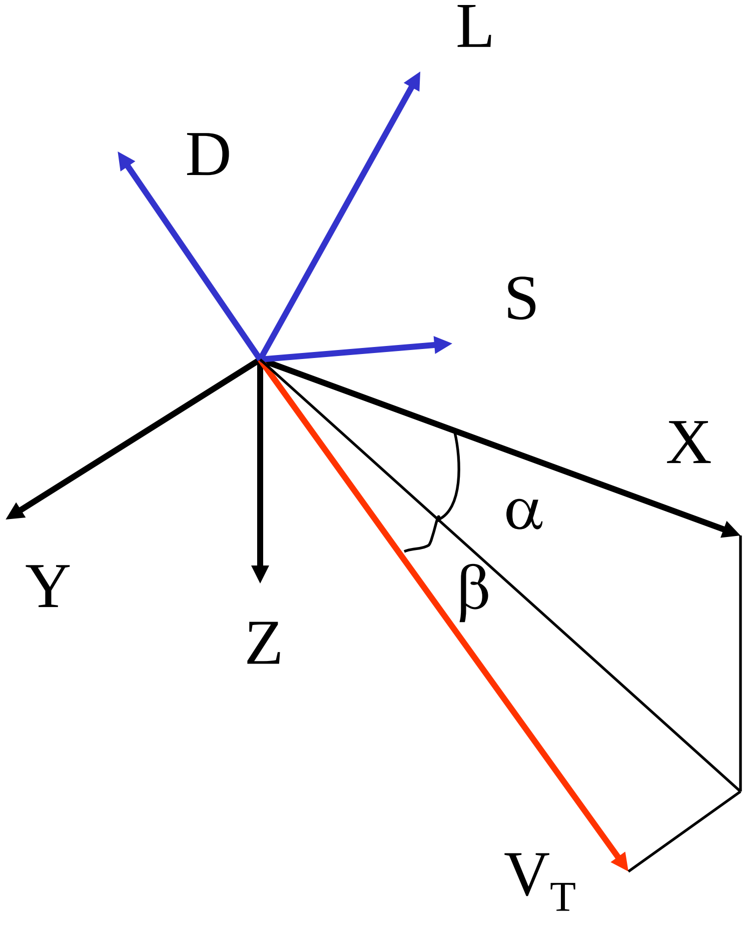
$$v = V_T \sin\beta$$

$$w = V_T \cos\beta \sin\alpha$$

Question : What about the aerodynamic forces ?

3.2 Writing aerodynamic forces in the body axes

Often aerodynamic forces are specified in terms of 3 mutually perpendicular forces



D : drag, aero force opposite to $\mathbf{V_T}$

L : lift, aero force perpendicular to $\mathbf{V_T}$

S : side force

NB : L, D & S defines an axes system i.e. the "flight path axes"

The transformation can be written in terms of two rotations

1) a rotation about the Y body axes by $-\alpha$

2) a rotation about the resulting Z axis by β

and then inverting the components

$$\begin{Bmatrix} D \\ S \\ L \end{Bmatrix} = - \begin{pmatrix} \cos(\beta) & \sin(\beta) & 0 \\ -\sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(-\alpha) & 0 & -\sin(-\alpha) \\ 0 & 1 & 0 \\ \sin(-\alpha) & 0 & \cos(-\alpha) \end{pmatrix} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}$$

Multiplying the rotation matrices yields the body axes components for the aerodynamic forces

$$D = -(X\cos\alpha + Z\sin\alpha)\cos\beta - Y\sin\beta$$

$$S = (X\cos\alpha + Z\sin\alpha)\sin\beta - Y\cos\beta$$

$$L = X\sin\alpha - Z\cos\alpha$$

or the inverse relation

$$X = L\sin\alpha + (S\sin\beta - D\cos\beta)\cos\alpha$$

$$Y = -(S\cos\beta + D\sin\beta)$$

$$Z = -L\cos\alpha + (S\sin\beta - D\cos\beta)\sin\alpha$$

Question : What do you expect if α and β are small ?