Lecture 8 : Dynamic Stability

Or what happens to small disturbances about a trim condition

1.0 : Dynamic Stability

Static stability refers to the tendency of the aircraft to counter a disturbance.

Dynamic stability is concerned with how fast the aircraft returns to trim condition after a disturbance.

Note that

1.1 Quantifying dynamic stability

The 6 DOF flight dynamics is a set of 9 ODEs of the form

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

e.g.

$\mathbf{x} = \{ u, v, w, p, q, r, \phi, \theta, \psi \}^T$ are the *states*

or $\{ \alpha, \beta, V_T, p, q, r, \phi, \theta, \psi \}^T$

and $\mathbf{u} = \{\delta a, \delta e, \delta r, \delta\pi \}^T$ are the *controls*

The dynamic stability analysis procedure is as follows:

Step 1. Find the trim point $\mathbf{x_o}$, $\mathbf{u_o}$ such that $d\mathbf{x}/dt = \mathbf{0}$ or

solve $\mathbf{f}(\mathbf{x}, \mathbf{u}) = \mathbf{0}$

Step 2. Consider a small disturbance about the trim condition,
i.e. set $\mathbf{x} = \mathbf{x_o} + \Delta\mathbf{x}$, $\mathbf{u} = \mathbf{u_o} + \Delta\mathbf{u}$

Expanding the ODEs about the trim condition

$$d(\mathbf{x}_o + \Delta\mathbf{x})/dt = \mathbf{f}(\mathbf{x_o}, \mathbf{u_o}) + [\,\partial\mathbf{f}/\partial\mathbf{x}\,]\,\Delta\mathbf{x} + [\,\partial\mathbf{f}/\partial\mathbf{u}\,]\,\Delta\mathbf{u} + \ldots.$$

For the trim point $d\mathbf{x}_o/dt = \mathbf{0}$, $\mathbf{f}(\mathbf{x}_o, \mathbf{u}_o) = \mathbf{0}$

Ignore higher order terms, the disturbance is governed by a linear system of ODEs of the form:

$$\Delta\mathbf{x}' = [\,A\,]\,\Delta\mathbf{x} + [\,B\,]\,\Delta\mathbf{u}$$

Step 3. Check the asymptotic stability of this linear system

i.e. How fast will the disturbance decrease to zero ?

1.2 Basic trim condition – steady, symmetric, level flight

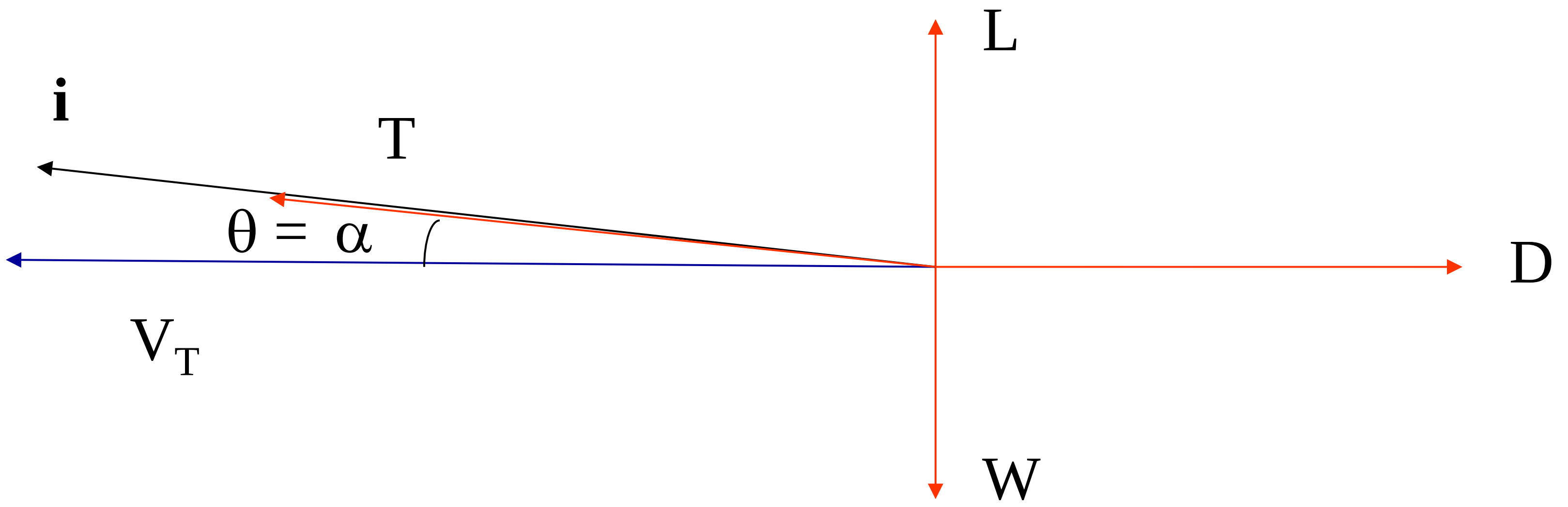
The basic trim condition is characterized by

steady flight - no rotational rates $p, q, r = 0$

symmetric flight – no sideslip $\beta, \delta r = 0$

wings level – no banking $\phi, \delta a = 0$

In this case the trim equations simplify to :



vertical forces

horizontal forces

pitching moment

Written in terms of aerodynamic coefficients, e.g. ...

$$\tfrac{1}{2}\,\rho(h)\,V^2 S\;\; C_L(\alpha, \delta e) \;+\; T(h, V, \delta\pi)\,\sin\alpha \qquad = \qquad W$$

$$\tfrac{1}{2}\,\rho(h)\,V^2 S\;\; C_D(\alpha, \delta e) \;-\; T(h, V, \delta\pi)\cos\alpha \;- \qquad = \qquad 0$$

$$C_m(\alpha, \delta e) \qquad = \qquad 0$$

Any observations ?

Question 1: How many equations and how many unknowns are there ?

Answer :

Question 2 : Which "unknowns" need to be specified ?

Answer :

Question : What happens if we specify h and then solve for corresponding α, V, δe ?

Sample F5 1g flight envelope

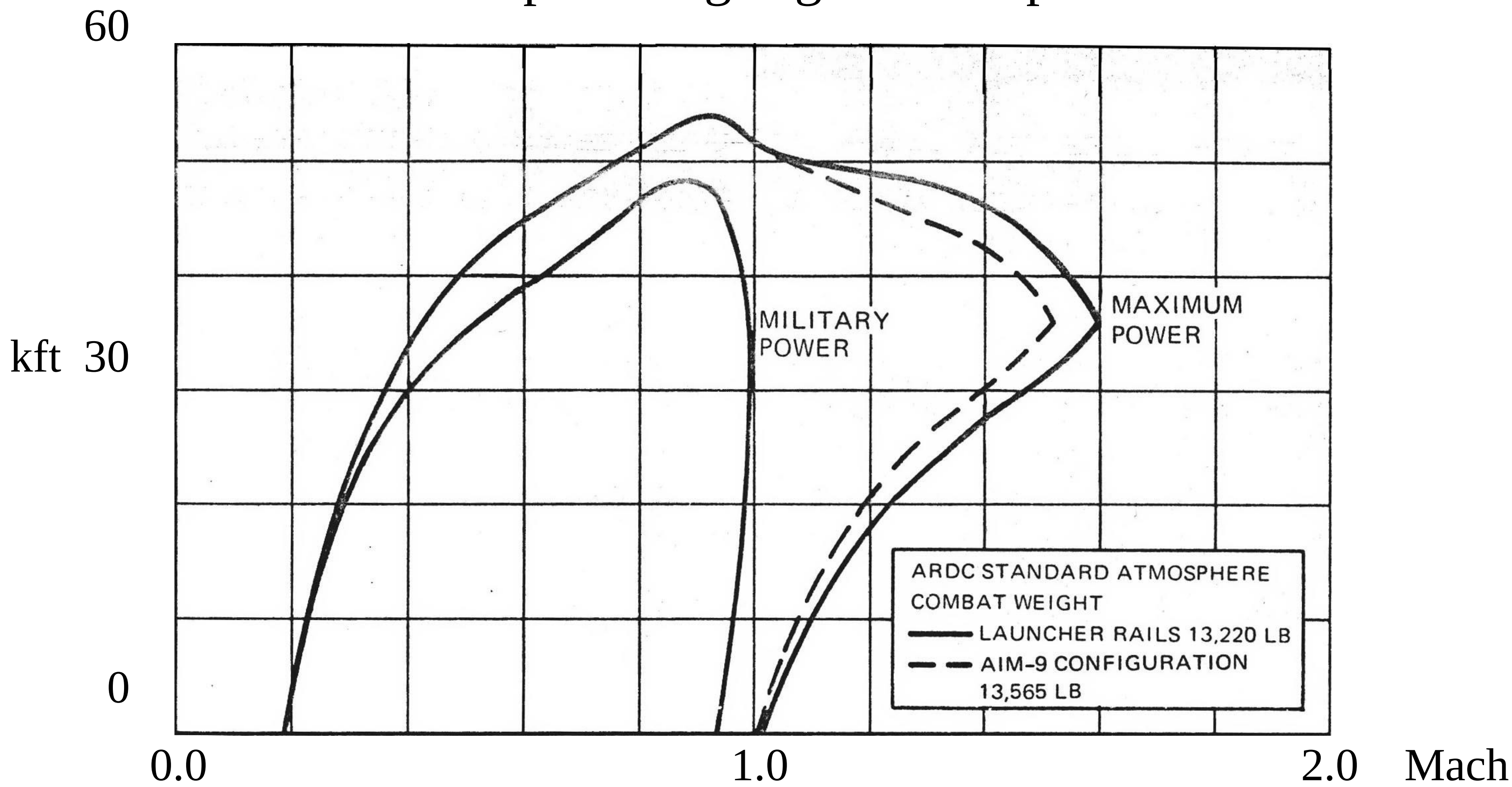
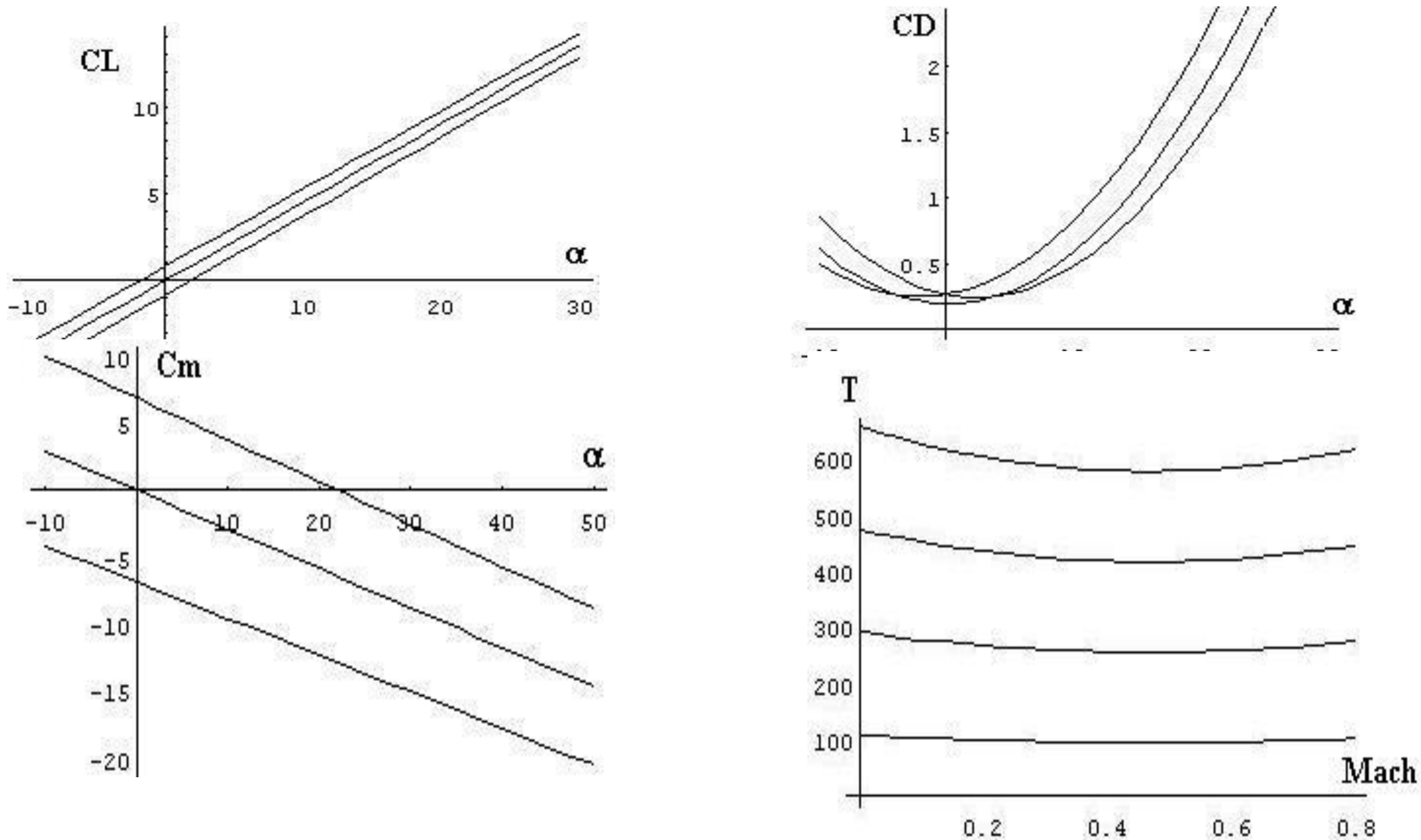


Figure 1.1: Typical aerodynamic & thrust characteristics



Note the "parabolic" nature of C_D

2.0 Longitudinal and lateral dynamic stability models

For a "moderate" range of α, β, p, q, r, the behaviour of the aerodynamics allow the separation of the EOM's into 2 decoupled sets

Longitudinal states & controls

$\mathbf{x} = \{\alpha, V_T, q, \theta\}$ or $\{w, u, q, \theta\}$ $\mathbf{u} = \{\delta e, \delta\pi\}$

i.e. the forward, up-down translation and pitching motion.

Lateral states & control

$\mathbf{x} = \{\beta, p, r, \phi\}$ or $\{v, p, r, \phi\}$ $\mathbf{u} = \{\delta a, \delta r\}$

i.e. the sideward, rolling and yawing motion.

This allows us to work with a smaller linear system when determining the stability of a trim point.

2.1 Math digression – Why stability is related to eigenvalues

1. To solve the linear system $\mathbf{x}' = [A]\,\mathbf{x}$, we look for solutions of the form $\mathbf{x} = \mathbf{v}\,e^{\lambda t}$ where $\mathbf{v}$ and λ are unknown.

2. This implies that $\lambda\,\mathbf{v}\,e^{\lambda t} = [A]\,\mathbf{v}\,e^{\lambda t}$

or $[A - \lambda I]\,\mathbf{v} = \mathbf{0}$

3. For non trivial solutions we require $\det|A - \lambda I| = 0$

4. This is the eigenvalue problem. Hence λ are given by the eigenvalues of A and $\mathbf{v}$ are the corresponding eigenvectors

2.3 Longitudinal linear (small disturbance) model

$$\mathbf{x}' = [A]\,\mathbf{x} + [B]\,\mathbf{u} \qquad (\Delta\text{'s omitted})$$

$$\mathbf{x} = \{\alpha,\ u/V_{To},\ q,\ \theta\}^T$$

$$\mathbf{u} = \{\delta e\}$$

$$[A] = \begin{pmatrix} Z_\alpha & Z_u & 1 + Z_q & 0 \\ X_\alpha & X_u & X_q & -g/V_{To} \\ m_\alpha & m_u & m_q & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad [B] = \begin{pmatrix} Z_\delta \\ X_\delta \\ m_\delta \\ 0 \end{pmatrix}$$

2.4 Lateral linear (small disturbance) model

$$\mathbf{x}' = [\,A\,]\,\mathbf{x} + [\,B\,]\,\mathbf{u} \qquad (\,\Delta\text{'s omitted})$$

$$\mathbf{x} = \{\beta,\ p,\ r,\ \phi\ \}^T$$

$$\mathbf{u} = \{\delta a,\ \delta r\}^T$$

$$[A] = \begin{pmatrix} Y_\beta & Y_p & Y_r - 1 & g/V_{To} \\ l_\beta & l_p & l_r & 0 \\ n_\beta & n_p & n_r & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad [B] = \begin{pmatrix} Y_{\delta a} & Y_{\delta r} \\ l_{\delta a} & l_{\delta r} \\ n_{\delta a} & n_{\delta r} \\ 0 & 0 \end{pmatrix}$$

The subscripted terms in the linear models are the control stability derivatives.

They are divided into

a) static derivatives eg : $Z_\alpha \; Z_u \; Z_\delta$

b) dynamic derivatives eg : Z_q

c) rate derivatives eg : $Z_{\alpha'}$

This is the starting point for the dynamic stability analysis

3.0 Dynamic stability analysis

Stability derivatives for aircraft trimmed at V_{To} = 660 ft/s (201 m/s)

Longitudinal stability derivatives

$Z_\alpha = 0.0016$	$Z_u = -0.105$	$Z_q = 0$
$X_\alpha = -1.43$	$X_u = -0.0955$	$X_q = 0$
$m_\alpha = -15.51$	$m_u = 0$	$m_q = -1.92$

Lateral stability derivatives

$Y_\beta = -0.0839$	$Y_p = 0$	$Y_r = 0$
$l_\beta = -4.5408$	$l_p = -1.699$	$l_r = 0.1717$
$n_\beta = 3.3792$	$n_p = -0.0654$	$n_r = -0.0893$

Longitudinal stability derivatives

$Z_\alpha = 0.0016$ $\quad Z_u = -0.105$ $\quad Z_q = 0$

$X_\alpha = -1.43$ $\quad X_u = -0.0955$ $\quad X_q = 0$

$m_\alpha = -15.51$ $\quad m_u = 0$ $\quad m_q = -1.92$

$$A = \begin{pmatrix} Z_\alpha & Z_u & 1 + Z_q & 0 \\ X_\alpha & X_u & X_q & -g/V_{To} \\ m_\alpha & m_u & m_q & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.0016 & -0.105 & 1 & 0 \\ -1.43 & -0.0955 & 0 & -0.0488 \\ -15.51 & 0 & -1.92 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Form the characteristic equation $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 0.0016 - \lambda & -0.105 & 1 & 0 \\ -1.43 & -0.0955 - \lambda & 0 & -0.0488 \\ -15.51 & 0 & -1.92 - \lambda & 0 \\ 0 & 0 & 1 & -\lambda \end{vmatrix}$$

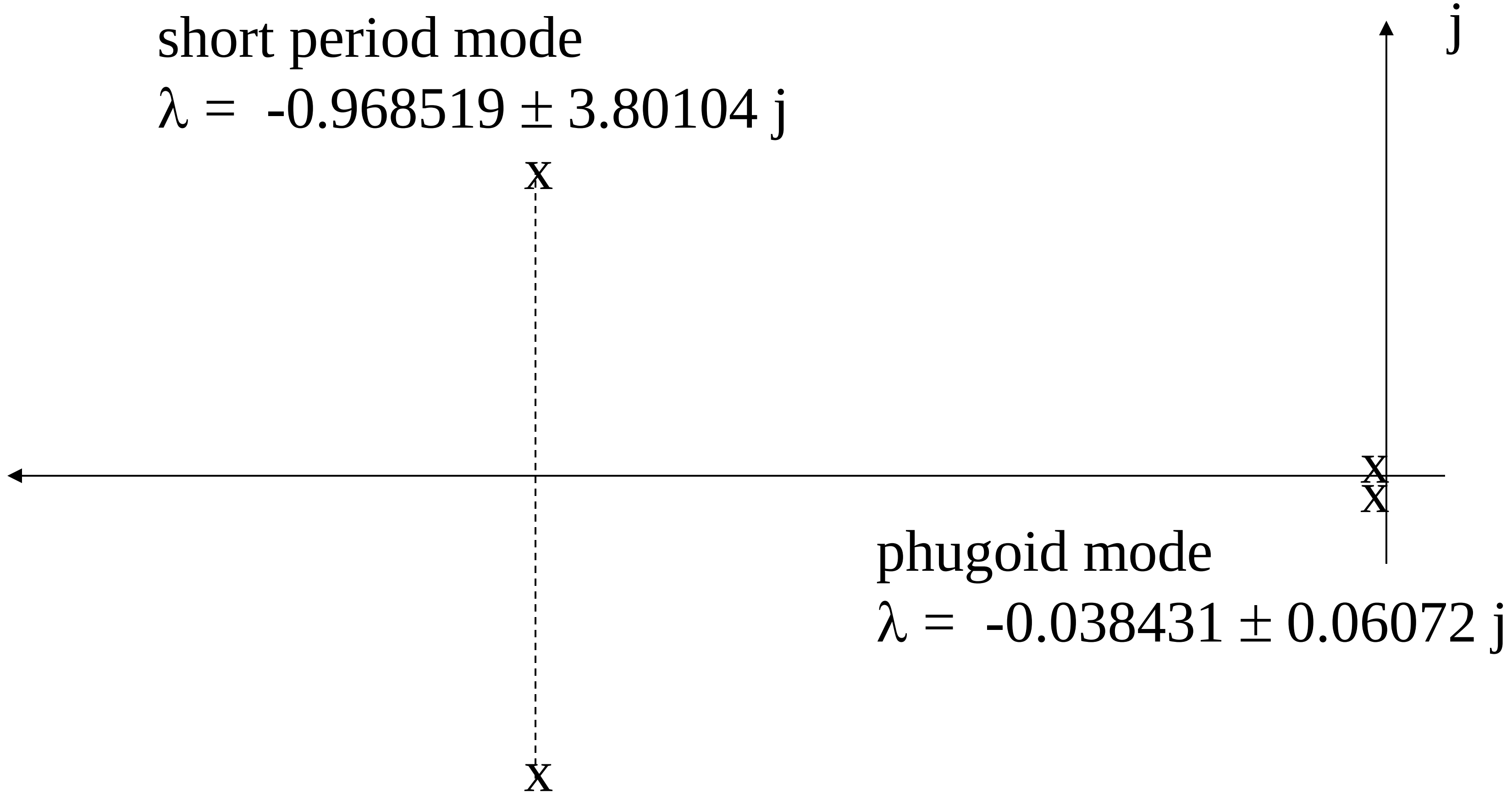
The characteristic equation is a quartic polynomial

$$0.0794732 + 1.19262\,\lambda + 15.54\,\lambda^2 + 2.0139\,\lambda^3 + \lambda^4 = 0$$

With the roots (eigenvalues)

$$\lambda = \begin{array}{l} -0.968519 \pm 3.80104\,j \\ -0.038431 \pm 0.06072\,j \end{array}$$

Plotting the eigenvalues in the complex plane



The eigenvector for each λ is obtained by solving $A\mathbf{v} = \lambda\mathbf{v}$

For the short period

$$\mathbf{v} = \left\{ \begin{array}{l} 0.384815 \pm 0.948234\,j \\ -0.317828 \pm 0.222808\,j \\ -4.01096 \pm 0.566191\,j \\ 0.392359 \pm 0.955254\,j \end{array} \right\}$$

For the phugoid

$$\mathbf{v} = \left\{ \begin{array}{l} 0.209759 \mp 0.16257\,j \\ -16.0527 \pm 13.0971\,j \\ -1.68406 \pm 1.39444\,j \\ 28.9253 \pm 9.42526\,j \end{array} \right\}$$

3.1 Interpreting the longitudinal eigenvalues & eigenvectors

Recall the disturbances are $\mathbf{x} = \{\alpha, u/V_{To}, q, \theta\}^T$

In terms of the eigenvalues and vectors, $\mathbf{x} = \Sigma C_i \mathbf{v} e^{\lambda t}$.
The contribution from the short period mode takes the form:

$$\begin{pmatrix} 1.02334 \\ 0.38814 \\ 4.05073 \\ 1.03269 \end{pmatrix} e^{-0.968519\, t} \{ \cos(3.80104t) \text{ or } \sin(3.80104t) \}$$

Now interpret the behaviour of the phugoid mode

$$\begin{pmatrix} 0.2655 \\ 20.7267 \\ 2.1874 \\ 30.4391 \end{pmatrix} e^{-0.038431t} \{ \cos(0.06072t) \text{ or } \sin(0.06072t) \}$$

Which states are strongly affected by the phugoid mode ?

Summary : Longitudinal modes

1. The longitudinal eigenvalues typically consists of two well separated complex pairs called the short period mode (stable) and the phugoid mode (marginally stable or unstable).

2. The short period mode is oscillatory and relatively well damped. It affects disturbances in α and q.

3. The phugoid mode is oscillatory and takes a long time to decay or grow. It affects disturbances in u and θ.

3.2 FAR 23.181 - Dynamic stability, paragraphs (a) & (d)

(a) Any short period oscillation not including combined lateral-directional oscillations occurring between the stalling speed and the maximum allowable speed appropriate to the configuration of the airplane must be heavily damped with the primary controls --
(1) Free; and
(2) In a fixed position

(d) …. Any long-period oscillation of flight path, phugoid oscillation, that results must not be so unstable as to increase the pilot's workload or otherwise endanger the airplane.

Lateral stability derivatives

$Y_\beta = -0.0839 \quad Y_p = 0 \quad Y_r = 0$

$l_\beta = -4.5408 \quad l_p = -1.699 \quad l_r = 0.1717$

$n_\beta = 3.3792 \quad n_p = -0.0654 \quad n_r = -0.0893$

$$[A] = \begin{pmatrix} Y_\beta & Y_p & Y_r - 1 & g/V_{To} \\ l_\beta & l_p & l_r & 0 \\ n_\beta & n_p & n_r & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -0.0839 & 0 & -1 & 0.0488 \\ -4.5408 & -1.699 & 0.1717 & 0 \\ 3.3792 & -0.0654 & -0.0893 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Similarly for the lateral eigenvalues and vectors

The eigenvalues are : -1.77973,
-0.0469122 ± 1.87764j
0.00135766

The lateral eigenvalues typically consists of a stable real eigenvalue (*roll*), a stable complex pair (*Dutch roll*) and a marginally stable or unstable real eigenvalue (*spiral*)

Homework : Find the eigenvectors

3.3 Interpreting the lateral eigenvalues & eigenvectors

$$\mathbf{x} = \{\beta, p, r, \phi\}^T$$

roll mode

$$\begin{pmatrix} 0.24480 \\ 13.6837 \\ 0.04003 \\ 7.68863 \end{pmatrix} e^{-1.77973t}$$

Dutch roll mode

$$\begin{pmatrix} 0.88108 \\ 1.60376 \\ 1.62547 \\ 0.85387 \end{pmatrix} e^{-0.0469122t} \{\cos(1.87764t) \text{ or } \sin(1.87764t)\}$$

spiral mode

$$\begin{pmatrix} 0.02356 \\ 0.02401 \\ 0.86083 \\ 17.6855 \end{pmatrix} e^{0.00135766t}$$

Summary : Lateral modes

1. The lateral eigenvalues typically consists of a stable real eigenvalue (roll), a stable complex pair (Dutch roll) and a marginally stable or unstable real eigenvalue (spiral).

2. The roll mode affects p and ϕ.

3. The Dutch roll is a coupled oscillatory roll-yaw motion (p and r).

4. The spiral mode affects mainly ϕ and hence r.

3.4 FAR Part 23. Section 181 - Dynamic stability, paragraph (b)

(b) Any combined lateral-directional oscillations ("Dutch roll") occurring between the stalling speed and the maximum allowable speed appropriate to the configuration of the airplane must be damped to 1/10 amplitude in 7 cycles with the primary controls --

(1) Free; and

(2) In a fixed position.

Question : What's the constraint on the eigenvalue ?

3.5 A comparison with FAR Part 25.181 – Dynamic Stability

(a) Any short period oscillation, not including combined lateral-directional oscillations, occurring between 1.13 VSR and maximum allowable speed appropriate to the configuration of the airplane must be heavily damped with the primary controls

(1) Free; and

(2) In a fixed position.

(b) Any combined lateral-directional oscillations ("Dutch roll") occurring between 1.2 VS and maximum allowable speed appropriate to the configuration of the airplane must be positively damped with controls free, and must be controllable with normal use of the primary controls without requiring exceptional pilot skill.