Simple Example of Stability Analysis

1. Particle of mass $m$ attached to a massless rigid rod of length $L$.

2. Rod is free to rotate about $O$.

3. Equation of motion -

$$mL^2 \theta'' = -mg L \sin \theta$$

So how do we start the stability analysis?
Step 0: Reduce to standard form

Standard form - A set of first order differential equations

For example,

Given \( mL^2 \theta'' = -mg L \sin \theta \)

Let \( \theta = x \) and \( \theta' = y \)

Hence

\[
\begin{align*}
x' &= y \\
y' &= - \left( \frac{g}{L} \right) \sin(x)
\end{align*}
\]
Step 1: Find the fixed points

Fixed points – Points \( \{x_0, y_0\} \) where \( x' = y' = 0 \)

Since \( x' = y \)

\[ y' = -\left(\frac{g}{L}\right) \sin(x) \]

The fixed points are given by :

\[ x' = 0 \implies y_0 = 0 \]

\[ y' = 0 \implies \sin(x_0) = 0 \implies x_0 = 0 \text{ or } \pi \]

So the fixed points are \( \{x_0, y_0\} = \{0,0\} \text{ and } \{\pi, 0\} \)
Step 2: Consider small perturbations about the fixed point

Let \( x = x_0 + dx \) \( y = y_0 + dy \)

Substitute in the equations of motion. Hence

\[
\begin{align*}
(x_0 + \ dx)' &= (y_0 + \ dy) = 0 \\
(y_0 + \ dy)' &= -(g/L) \sin(x_0 + dx) = 0 \\
&= -(g/L) \{ \sin(x_0) + \cos(x_0)dx + O(dx^2) \}
\end{align*}
\]

Equations for perturbations

\[
\begin{align*}
dx' &= dy \\
\ dy' &= -(g/L) \cos(x_0) \ dx
\end{align*}
\]
Step 3 : Check if perturbations grow with time - fixed point 1

For the fixed point \( \{x_0, y_0\} = \{0,0\} \), the small perturbation equations are:

\[
\begin{align*}
    dx' &= dy = 1 \\
    dy' &= -(g/L) \cos(x_0) dx
\end{align*}
\]

Guess a solution \( dx = C \exp(\lambda t) \), this implies that \( \lambda^2 = -(g/L) \)

or \( dx = C \exp \left[ -j (g/L)^{1/2} t \right] \) or \( C \exp \left[ +j (g/L)^{1/2} t \right] \)

So how does the perturbation behaves with time?

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Step 3 : Check if perturbations grow with time – fixed point 2

For the fixed point \( \{x_0, y_0\} = \{\pi, 0\} \), the small perturbation equations are:

\[
\begin{align*}
    dx' &= \quad dy = -1 \\
    dy' &= - (g/L) \cos(x_0) \, dx
\end{align*}
\]

Guess a solution \( dx = C \exp(\lambda t) \), this implies that \( \lambda^2 = \frac{g}{L} \)

i.e.

\[
    dx = C \exp\left[- (g/L)^{1/2} \, t \right] \quad \text{or} \quad C \exp\left[ + (g/L)^{1/2} \, t \right]
\]

So how does the perturbation behaves with time ?

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Example: The return of the pendulum

Recall the small perturbation model for the pendulum example

\[
\begin{align*}
\begin{bmatrix} dx' \\ dy' \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -(g/L) \cos(x_0) & 0 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \\
A &= \begin{bmatrix} 0 & 1 \\ -(g/L) \cos(x_0) & 0 \end{bmatrix} \\
|A - \lambda I| &= \lambda^2 + (g/L) \cos(x_0) = 0
\end{align*}
\]

Eigenvalues are \( \lambda = \pm \left[ - (g/L) \cos(x_0) \right]^{1/2} \)

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