Lecture 7 : Flight Equations of Motion

*Or the differential equations for a 6 DOF model*
1.0 Recap - 6 DOF Dynamics Model

- For flight dynamics & control, the reference frame is located at the cg, aligned with the aircraft and moves with it.
The 6 DOFs correspond to:

- **translation** along the X, Y, Z directions
  - velocity vector in body axes \( \mathbf{V} = \{u, v, w\} \)

- **rotation** about the X, Y, Z axes
  - angular velocity vector in body axes \( \mathbf{\omega} = \{p, q, r\} \)

Now we need to relate forces and moments to these 6 DOFs
• Denote the forces acting on the aircraft by $F = \{X, Y, Z\}$
  – These forces act at the origin (c.g) of the reference frame

• Similarly denote the moments by $M_{cg} = \{l, m, n\}$
  – These are the roll, pitch and yaw moments taken wrt the c.g.

Question: So how do we relate the force to the 3 translational DOF’s?

Answer: $F = \frac{d(mV)}{dt} = m \{u', v', w'\}$?
2.0 Coriolis theorem

Coriolis Theorem states that the absolute or actual time rate of change of a vector \( \mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \) where the reference frame \( \mathbf{ijk} \) rotates with absolute angular velocity \( \omega = p \mathbf{i} + q \mathbf{j} + r \mathbf{k} \) is:

\[
\frac{d\mathbf{r}}{dt} = \left( \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} \right) + \omega \times \mathbf{r}
\]
2.1 Translational equations of motion

The translational motion is governed by:

\[ \frac{\partial (mV)}{\partial t} + \omega \times (mV) = F \]

Assuming the mass \( m \) is constant over the time interval of interest, this works out as ...
Translational equations of motion

\[ u' + qw - rv = \frac{X}{m} \]
\[ v' + ru - pw = \frac{Y}{m} \]
\[ w' + pv - qu = \frac{Z}{m} \]

Note the nonlinear terms arising from the correction \( \omega x (mV) \)
Notes:

We can decompose

\[ \mathbf{F} = \mathbf{F}_{\text{aero}} + \mathbf{T} + \mathbf{W} \]

i.e.

aero forces \( \mathbf{F}_{\text{aero}} = \{X_{\text{aero}}, Y_{\text{aero}}, Z_{\text{aero}}\} \)

thrust \( \mathbf{T} = \{T_x, T_Y, T_Z\} \)

weight \( \mathbf{W} = Mg \{ -\sin\theta \quad \cos\theta \sin\phi \quad \cos\theta \cos\phi \} \)
3.0 Angular momentum and moments

The moments acting on the aircraft may be related to the 3 rotational DOFs by

\[ \frac{dH}{dt} = M \]

where \( M \) is the moment of the forces acting on the aircraft

\( H \) is the angular momentum of the aircraft

Question : Where’s the catch ?
The relation holds only if both $\mathbf{M}$ and $\mathbf{H}$ are taken wrt (a) or (b). The second case holds for our reference frame. So what’s $\mathbf{H}_{cg}$ for an aircraft?
3.1 Computing the angular momentum $H_{cg}$ of an aircraft

Consider a small piece of the aircraft of mass $dm$ located at $r = \{x,y,z\}$ wrt the body axes.
The linear momentum of this small mass relative to the cg is

\[ dL_{cg} = (\omega \times r) \ dm \]

\[ = \{p, q, r\} \times \{x, y, z\} \ dm \]

\[ = \{qz - ry, rx - pz, py - qx\} \ dm \]
The angular momentum wrt the cg of this small mass is

\[ dH_{cg} = r \times dL_{cg} \]

\[ = r \times (\omega \times r) \, dm \]

\[ = \{ x, y, z \} \times \{ qz - ry, rx - pz, py - qx \} \, dm \]

Finally we sum over the whole aircraft

\[ H_{cg} = \int \{ x, y, z \} \times \{ qz - ry, rx - pz, py - qx \} \, dm \]
Simplifying the rhs, the aircraft angular momentum is \( \mathbf{H}_{cg} = \mathbf{\omega} \mathbf{I}_{cg} \)

Where

\[
\mathbf{I}_{cg} = \begin{pmatrix}
\int (y^2 + z^2) \, dm & - \int xy \, dm & - \int zx \, dm \\
- \int xy \, dm & \int (z^2 + x^2) \, dm & - \int yz \, dm \\
- \int zx \, dm & - \int yz \, dm & \int (x^2 + y^2) \, dm
\end{pmatrix}
\]

is the moment of inertia matrix.

G. Leng, Flight Dynamics, Stability & Control
The *principal* moments of inertia are:

\[
I_{xx} = \int (y^2 + z^2) \, dm \quad I_{yy} = \int (z^2 + x^2) \, dm \quad I_{zz} = \int (x^2 + y^2) \, dm
\]

The *crossed* moments of inertia are:

\[
I_{xy} = \int xy \, dm \quad I_{yz} = \int yz \, dm \quad I_{zx} = \int zx \, dm
\]

What’s the difference with planar motion of rigid bodies?
Ex: Simplifications for symmetric aircraft

Question: Which is the plane of symmetry?

G. Leng, Flight Dynamics, Stability & Control
Consider \( I_{xy} = \int xy \, dm \)

For each section along the X axis, the contribution from the left side cancels the contribution from the right.

\[ \therefore \quad I_{xy} = 0 \quad \text{Is that all?} \]
Similarly for $I_{yz} = \int yz \, dm$

For each section along the $Z$ axis, the contribution from the left side cancels the contribution from the right.

$\therefore \quad I_{yz} = 0$
Hence the moment of inertia matrix simplifies to:

\[
I_{cg} = \begin{pmatrix}
I_{xx} & 0 & -I_{xz} \\
0 & I_{yy} & 0 \\
-I_{xz} & 0 & I_{zz}
\end{pmatrix}
\]
3.2 Rotational equations of motion

Now apply Coriolis theorem to \( \frac{dH_{cg}}{dt} = M_{cg} \)

where \( H_{cg} = \omega I_{cg} \) the rotational equations of motion are:

\[
\frac{\partial (\omega I_{cg})}{\partial t} + \omega \times (\omega I_{cg}) = M_{cg}
\]

Which simplifies \( (I_{xy} = I_{yz} = 0) \) to:

G. Leng, Flight Dynamics, Stability & Control
\[
\begin{align*}
I_{xx} p' + (I_{zz} - I_{yy}) qr - I_{xz} (r' + pq) &= 1 \\
I_{yy} q' + (I_{xx} - I_{zz}) pr - I_{xz} (r^2 - p^2) &= m \\
I_{zz} r' + (I_{yy} - I_{xx}) pq - I_{xz} (p' - qr) &= n
\end{align*}
\]

Assuming the XZ plane is a plane of symmetry
Summary: The basic 6 DOF model for symmetric aircraft

\[\begin{align*}
    u' + q w - r v &= (X_{aero} + T_x)/m - g \sin \theta \\
v' + r u - p w &= (Y_{aero} + T_y)/m + g \cos \theta \sin \phi \\
w' + p v - q u &= (Z_{aero} + T_z)/m + g \cos \theta \cos \phi \\
\end{align*}\]

\[\begin{align*}
    I_{xx} p' + (I_{zz} - I_{yy}) qr - I_{xz} (r' + pq) &= l \\
    I_{yy} q' + (I_{xx} - I_{zz}) pr - I_{xz} (r^2 - p^2) &= m \\
    I_{zz} r' + (I_{yy} - I_{xx}) pq - I_{xz} (p' - qr) &= n \\
\end{align*}\]

\[\begin{align*}
    \phi' &= p + (q \sin \phi + r \cos \phi) \tan \theta \\
    \theta' &= q \cos \phi - r \sin \phi \\
    \psi' &= (q \sin \phi + r \cos \phi) \sec \theta \\
\end{align*}\]
Homework

Assume the thrust vector lies in the plane of symmetry, act at the c.g. with an orientation angle $\varepsilon$ from the X axis as shown.

Using the relations:

$u = V_T \cos \beta \cos \alpha$

$v = V_T \sin \beta$

$w = V_T \cos \beta \sin \alpha$

Derive expressions for $\alpha'$, $\beta'$ and $V_T'$
Answer: These are the translational EOM in flight path axes

\[
\alpha' = q - (r \sin \alpha + p \cos \alpha) \tan \beta - \frac{(T \sin(\alpha + \varepsilon) + L)}{m V_T \cos \beta} + \frac{g W_1}{V_T \cos \beta}
\]

\[
\beta' = p \sin \alpha - r \cos \alpha - \frac{(T \cos(\alpha + \varepsilon) \sin \beta + S)}{m V_T} + \frac{g W_2}{V_T}
\]

\[
V_T' = \frac{(T \cos(\alpha + \varepsilon) \cos \beta - D)}{m} - g W_3
\]

where

\[
W_1 = \cos(\theta - \alpha) - (1 - \cos \phi) \cos \theta \cos \alpha
\]

\[
W_2 = \sin(\theta - \alpha) \sin \beta + \cos \theta \sin \phi \cos \beta + (1 - \cos \phi) \cos \theta \sin \alpha \sin \beta
\]

\[
W_3 = \sin(\theta - \alpha) \cos \beta - \cos \theta \sin \phi \sin \beta + (1 - \cos \phi) \cos \theta \sin \alpha \cos \beta
\]

G. Leng, Flight Dynamics, Stability & Control